

Very cost effective bipartition in $\Gamma(\mathbb{Z}_n)$

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ABSTRACT. Let \mathbb{Z}_n be the finite commutative ring of residue classes *modulo* n and $\Gamma(\mathbb{Z}_n)$ be its zero-divisor graph. The nilradical graph and non-nilradical graph of \mathbb{Z}_n are denoted by $N(\mathbb{Z}_n)$ and $\Omega(\mathbb{Z}_n)$ respectively. In 2012, Haynes et al. [5] introduced the concept of very cost effective graph. For a graph $G = (V, E)$ and a set of vertices $S \subseteq V$, a vertex $v \in S$ is said to be very cost effective if it is adjacent to more vertices in $V \setminus S$ than in S . A bipartition $\pi = \{S, V \setminus S\}$ is called very cost effective if both S and $V \setminus S$ are very cost effective sets [5, 6]. In this paper, we investigate the very cost effective bipartition of $\Gamma(\mathbb{Z}_n)$, where $n = p_1 p_2 \cdots p_m$, here all p_i 's are distinct primes. In addition, we discuss the cases in which $N(\mathbb{Z}_n)$ and $\Omega(\mathbb{Z}_n)$ graphs have very cost effective bipartition for different n . Finally, we derive some results for very cost effective bipartition of the Line graph and Total graph of $\Gamma(\mathbb{Z}_n)$, denoted by $L(\Gamma(\mathbb{Z}_n))$ and $T(\Gamma(\mathbb{Z}_n))$ respectively.

1. INTRODUCTION

Let $R = \mathbb{Z}_n$ be the finite commutative ring of residue classes *modulo* n with identity ($1 \neq 0$) and $\Gamma(\mathbb{Z}_n)$ be its zero-divisor graph. The study of zero-divisor graphs of commutative rings reveals interesting relation between ring theory and graph theory. Because, algebraic tools help to understand graphs properties and vice versa. An element $z (\neq 0) \in R$ is said to be a zero-divisor if there exists a non-zero $r \in R$ such that $rz = 0$. The set of zero-divisors is denoted by $Z(R)$ and zero-divisor graph of R , $\Gamma(R)$, is the graph whose vertices are the zero-divisors of R and its two vertices are connected by an edge if and only if their product is 0. In 1988, the concept of zero-divisor graph of a commutative ring was introduced by I. Beck [2] in context of coloring of rings and were redefined in 1999 by D. F Anderson and P. Livingston in [1]. For a graph $G = (V, E)$, the open neighborhood of a vertex $u \in V$ is the set

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$N(u) = \{v \mid uv \in E\}$, and the closed neighborhood of u is the set $N[u] = N(u) \cup \{u\}$. In the same fashion, the open neighborhood of a set $S \subseteq V$ is the set $N(S) = \cup_{u \in S} N(u)$, and the closed neighborhood is the set $N[S] = N(S) \cup S$ respectively. The order of the open neighborhood of a vertex $u \in V$ is denoted by $|N(u)|$. The degree of a vertex v in a graph G , denoted by $\deg(v)$, is $|N(v)|$. For basic definitions and results on graph we refer [4].

In 2012, the concept of cost effective and very cost effective sets in graphs were introduced by Haynes et al.[5] and further studied in [6] for various graphs. A vertex v in a set S is said to be cost effective if it is adjacent to at least as many vertices in $V \setminus S$ as in S , that is, $|N(v) \cap S| \leq |N(v) \cap V \setminus S|$. A Vertex v in a set S is very cost effective if it is adjacent to more vertices in $V \setminus S$ than in S , that is, $|N(v) \cap S| < |N(v) \cap V \setminus S|$. A set S is (very) cost effective if every vertex $v \in S$ is (very) cost effective. Moreover, very cost effective bipartition were also introduced in [5]. A bipartition $\pi = \{S, V \setminus S\}$ is called cost effective if each of S and $V \setminus S$ is cost effective, and π is very cost effective if each of S and $V \setminus S$ is very cost effective. Graphs that have a (very) cost effective bipartition are called (very) cost effective graphs. It was shown in [5] that every connected, non-trivial graph is cost effective. Also, they observed in [6] that all bipartite graphs with no isolated vertices are very cost effective.

A line graph $L(G)$ of a simple graph G is obtained by associating a vertex with each edge of the graph G and connecting two vertices by an edge if and only if the corresponding edges of G have a vertex in common. The total graph $T(G)$ of the graph G has a vertex for each edge and each vertex of G and an edge in $T(G)$ for every edge-edge, vertex-edge, and vertex-vertex adjacency in G . Here, we illustrate an example of a Zero divisor graph and its line graph over ring \mathbb{Z}_{16} :

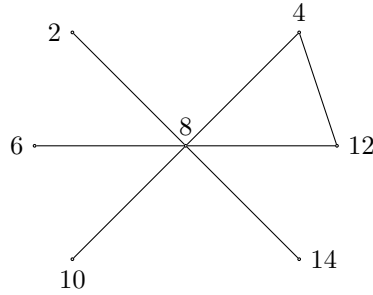
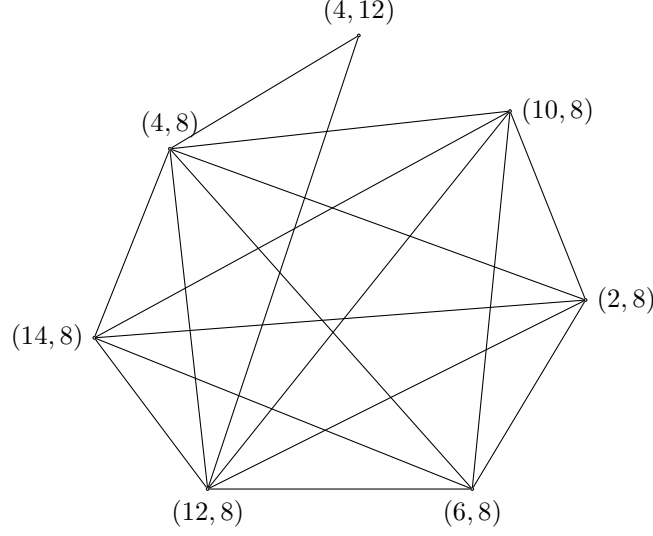


Figure 1 $\Gamma(\mathbb{Z}_{16})$


 Figure 2 $L(\Gamma(\mathbb{Z}_{16}))$

2. Bipartition in $\Gamma(\mathbb{Z}_n)$ and $L(\Gamma(\mathbb{Z}_n))$

Let $\pi = \{R, B\}$ be a bipartition of the graph G . If π is a very cost effective bipartition of G , then we say that G is very cost effective under π .

THEOREM 2.1. *If $n = p_1 \cdot p_2 \cdots p_m$, $m \geq 1$ and $p_1 < p_2 < \cdots < p_m$ are primes, then $\Gamma(\mathbb{Z}_n)$ is a very cost effective graph.*

PROOF. Let $n = p_1 \cdot p_2 \cdots p_m$, $m \geq 1$ for distinct primes p_1, p_2, \dots, p_m . Then all the zero divisor elements of \mathbb{Z}_n are $p_1, 2p_1, \dots, (p_2 \cdots p_m - 1)p_1; p_2, 2p_2, \dots, (p_1 p_3 \cdots p_m - 1)p_2; \dots, p_m, 2p_m, \dots, (p_1 p_2 \cdots p_{m-1} - 1)p_m$. Let $\pi = \{R, B\}$ be a bipartition of vertices in $\Gamma(\mathbb{Z}_n)$. Suppose the set R contains all the elements which are multiple of p_m and the other set B contains rest of elements. Then B is an independent set because it does not contain any element which is multiple of p_m . Now, we take $u \in B$. Then there exists at least one element $v \in R$ such that $uv = 0$. Therefore, $|N(u) \cap B| = 0$ and $|N(u) \cap R| \geq 1$, for all $u \in B$. Hence, all the elements in set B are very cost effective and thus the set B is very cost effective set.

Now, we divide R into two sets R_1 and R_2 . R_1 is the subset of R containing multiple of p_m as well as some of p_i 's (not all at a time), for $i = 1, \dots, m - 1$ and R_2 is containing those multiples of p_m which are not multiple of any p_i 's, $i = 1, \dots, m - 1$. Then element $v \in R_2$ is not adjacent to any element in R . But, these elements are adjacent to the elements of set B , so $|N(v) \cap R| = 0$ and

$|N(v) \cap B| \geq 1$. Again, if $u \in R_1$, then number of vertices adjacent to u in R is $\lfloor \frac{p_1 \cdot p_2 \cdots p_{m-1} - 1}{\prod p_i} \rfloor = M$, where $\prod p_i$ is the product of those p_i 's $i = 1, \dots, m-1$ which are not available in u and number of vertices which are adjacent to u in B is $((\prod p_i)p_m - 1 - M)$ where $\prod p_i$ is the product of those p_i 's $i = 1, \dots, m-1$ which are available in u . Since $|N(u) \cap R| = M$ and $|N(u) \cap B| > M$. So $|N(u) \cap R| < |N(u) \cap B|$. Therefore, the set B is also very cost effective and the partition $\pi = \{R, B\}$ is very cost effective bipartition. Hence, the graph $\Gamma(\mathbb{Z}_n)$ is very cost effective graph. \square

COROLLARY 2.2. $\Gamma(\mathbb{Z}_n)$ is very cost effective graph, for $n = pq$, where p, q are distinct primes.

PROOF. For $n = pq$, there are two independent sets of vertices in $\Gamma(\mathbb{Z}_n)$. One set contains multiple of p and other multiple of q respectively. Therefore, the graph $\Gamma(\mathbb{Z}_n)$ is a complete bipartite graph. Since every bipartite graph without isolated vertices is very cost effective. Thus, $\Gamma(\mathbb{Z}_n)$ is very cost effective graph. \square

THEOREM 2.3. Let p and q be distinct primes and n a positive integer.

- (i) If $n = p^2q$, $p, q \geq 2$, then $\Gamma(\mathbb{Z}_n)$ is very cost effective.
- (ii) If $n = p^2q^2$, $p, q \geq 3$ and $p < q$, then $\Gamma(\mathbb{Z}_n)$ is very cost effective.

PROOF. (i) Let $n = p^2q$, where p, q be distinct primes. Certainly, zero divisor elements of \mathbb{Z}_n are either multiples of p or multiples of q or multiples of pq . Let R be the set of vertices contains all those elements which are multiple of q and B contains all those elements which are multiple of p but not q . Let $\pi = \{R, B\}$ be the bipartition of $V(\Gamma(\mathbb{Z}_n))$. Then B is an independent set. Now, take $u \in B$, then $|N(u) \cap B| = 0$ and $|N(u) \cap R| \geq 1$. In the set R , take $v \in R$. If v is multiple of pq , then $|N(v) \cap R| = p-2$ and $|N(v) \cap B| > p-2$. Again, if v is a multiple of q only, then $|N(v) \cap R| = 0$ and $|N(v) \cap B| \geq 1$. Hence, from both the condition $|N(v) \cap R| < |N(v) \cap B|$. Therefore, the sets R and B are very cost effective and thus, the graph $\Gamma(\mathbb{Z}_n)$ is very cost effective.

(ii) Let $n = p^2q^2$, where p, q are distinct odd primes. Then the zero divisor elements are either multiple of p or q or both. Now, take a bipartition $\pi = \{R, B\}$ in such a way that set $R = R_1 \cup R_2 \cup R_3$ where $R_1 = \{v \in V(\Gamma(\mathbb{Z}_{p^2q^2})) : p^2 \mid v\}$, $R_2 = \{v \in V(\Gamma(\mathbb{Z}_{p^2q^2})) : p \mid v \text{ and } q \nmid v\}$ and $R_3 = \{v \in V(\Gamma(\mathbb{Z}_{p^2q^2})) : pq \mid v \text{ and } p^2 \nmid v \text{ and } q^2 \nmid v\}$. Here, R_3 contains $\frac{q(p-2)+1}{2}$ number of vertices. Now, set $B = B_1 \cup B_2 \cup B_3$ where $B_1 = \{v \in V(\Gamma(\mathbb{Z}_{p^2q^2})) : q^2 \mid v\}$, $B_2 = \{v \in V(\Gamma(\mathbb{Z}_{p^2q^2})) : q \mid v \text{ and } p \nmid v\}$ and $B_3 = \{v \in V(\Gamma(\mathbb{Z}_{p^2q^2})) : pq \mid v \text{ and } p^2 \nmid v \text{ and } q^2 \nmid v\}$. Here, B_3 contains $\frac{p(q-2)+1}{2}$ number of vertices. Let $u \in R$. If u is not a multiple of q , then $|N(u) \cap R| = 0$ and $|N(u) \cap B| \geq 1$. Again, if u is a multiple of p as well as q , then $|N(u) \cap R| = \frac{pq-3}{2}$ and $|N(u) \cap B| \geq \frac{pq-1}{2}$.

So $|N(u) \cap R| < |N(u) \cap B|$ and the set R is very cost effective. Similarly, set B is also very cost effective and the bipartition π is very cost effective bipartition. Thus, the graph $\Gamma(\mathbb{Z}_n)$ is very cost effective. \square

THEOREM 2.4. *$L(\Gamma(\mathbb{Z}_n))$ is very cost effective, Where $n = pq$, $p < q$ and p, q are primes.*

PROOF. If $n = pq$, then zero divisor graph $\Gamma(\mathbb{Z}_n)$ is a complete bipartite graph. So, there are two independent sets of vertices in which each vertex of a set is adjacent to every vertex of the other set. We draw the line graph of $\Gamma(\mathbb{Z}_n)$ with $(p-1)(q-1)$ vertices. This line graph is $(p+q-4)$ regular graph. Let $[u_i, v_j] \in V(L(\Gamma(\mathbb{Z}_n)))$, where u_i 's are multiple of p i.e $u_i = p.i$, $1 \leq i \leq q-1$ and v_j 's are multiple of q i.e $v_j = q.j$, $1 \leq j \leq p-1$. Now, Let $\pi = \{R, B\}$ be a bipartition of vertices in $L(\Gamma(\mathbb{Z}_n))$. Now we shall prove that π is very cost effective bipartition. Since each set R and B contain $\frac{(p-1)(q-1)}{2}$ vertices. Therefore, set R contains $\{[u_1, v_k], [u_2, v_L], [u_3, v_k], [u_4, v_L], \dots, [u_{q-2}, v_k], [u_{q-1}, v_L]\}$ where $1 \leq k \leq \frac{p-1}{2}$ and $\frac{p+1}{2} \leq L \leq p-1$ and B contains $\{[u_1, v_L], [u_2, v_k], [u_3, v_L], [u_4, v_k], \dots, [u_{q-2}, v_L], [u_{q-1}, v_k]\}$ where $1 \leq k \leq \frac{p-1}{2}$ and $\frac{p+1}{2} \leq L \leq p-1$. Now, take $[u_i, v_j] \in R$. Then $|N([u_i, v_j]) \cap R| = \frac{p+q}{2} - 3$ and $|N([u_i, v_j]) \cap B| = \frac{p+q}{2} - 1$. Therefore, $|N([u_i, v_j]) \cap R| < |N([u_i, v_j]) \cap B|$. Hence, R is very cost effective set. Similarly, if we take $[u_i, v_j] \in B$, then $|N([u_i, v_j]) \cap B| < |N([u_i, v_j]) \cap R|$. Thus, the line graph $L(\Gamma(\mathbb{Z}_n))$ is very cost effective graph. \square

3. Bipartition in $N(\mathbb{Z}_n)$ and $\Omega(\mathbb{Z}_n)$

In 2008, Bishop et al. [3] introduced the concept of Nilradical graph and Non-Nilradical graph and further some work appeared in [7]. They defined these graphs as follows:

DEFINITION 3.1. [3] The nilradical graph, denoted $N(R)$, is the graph whose vertices are the nonzero nilpotent elements of R and two vertices are connected by an edge if and only if their product is 0.

DEFINITION 3.2. [3] The non-nilradical graph, denoted $\Omega(R)$, is the graph whose vertices are the non-nilpotent zero-divisors of R and where two vertices are connected by an edge if and only if their product is 0.

Fig 1 is serve an example of $N(\mathbb{Z}_{16})$. Also the example of $\Omega(\mathbb{Z}_{18})$ is given below:

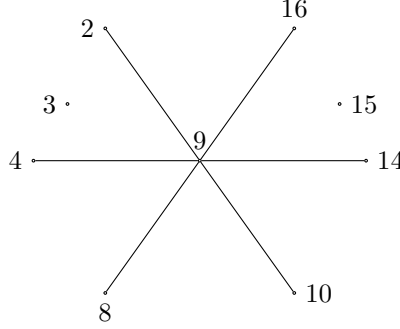


Figure 3 $\Omega(\mathbb{Z}_{18})$

THEOREM 3.3. *Let p, q be distinct primes and n , a positive integer.*

- (i) *If $n = p^2$ and $p > 2$, then $N(\mathbb{Z}_n)$ is very cost effective graph.*
- (ii) *If $n = p^2q^2$ and $p, q \geq 2$, then $N(\mathbb{Z}_n)$ is very cost effective graph.*
- (iii) *If $n = p^3$ and $p \geq 2$, then $N(\mathbb{Z}_n)$ is very cost effective graph.*
- (iv) *If $n = p^2q$ and $p > 2$, then $N(\mathbb{Z}_n)$ is very cost effective graph.*

PROOF. (i) Let $n = p^2$, where p is a prime number and $p > 2$. Then the nilpotent elements in \mathbb{Z}_n are $p, 2p, \dots, (p-1)p$. So, the number of nilpotent elements is $p-1$ and every element is adjacent to the other element. So, these $(p-1)$ elements forms a complete graph. Since p is prime, then $p-1$ is even and every complete graph of even order is very cost effective. Thus, $N(\mathbb{Z}_n)$ is very cost effective graph.

(ii) Let $n = p^2q^2$, where p, q are distinct primes. Then the nilpotent elements in \mathbb{Z}_n are multiple of pq and number of nilpotent elements are $pq-1$. Since all the nilpotent elements are multiple of pq , so every vertex is adjacent to the all other vertices in $N(\mathbb{Z}_n)$. Therefore, $(pq-1)$ elements form a complete graph with $pq-1$ vertices. Since p and q are prime number and $p, q > 2$, so $pq-1$ is even number. Hence, the graph of $N(\mathbb{Z}_n)$ is very cost effective.

(iii) If $n = p^3$, where p is prime number, then all the nilpotent elements are multiple of p and total number of nilpotent elements are p^2-1 . Now, we take bipartition $\pi = \{R, B\}$ of vertex set in $N(\mathbb{Z}_n)$ such that R contains only those elements which are divisible by p but not by p^2 and the set B contains those elements which are divisible by only p^2 . Then the set R has $p(p-1)$ elements and the set B has $p-1$ elements. Now, set R is independent set and in B , each vertex is adjacent to every other vertices in B . Since all the elements of R is adjacent to all elements of set B . Therefore, R is very cost effective set. Now, take $u \in B$, then $|N(u) \cap B| = p-2$ and $|N(u) \cap R| = p(p-1)$. Since $|N(u) \cap B| < |N(u) \cap R|$, so set B is very cost effective. Hence, $\pi = \{R, B\}$ is

very cost effective bipartition and graph $N(\mathbb{Z}_n)$ is very cost effective.

(iv) Let $n = p^2q$, where p and q are distinct prime number and $p \neq q$. Then the nilpotent elements of $N(\mathbb{Z}_n)$ are multiples of pq and the number of nilpotent elements are $p - 1$. These $p - 1$ elements are connected to each other so these $p - 1$ vertices forms a complete graph. Since p is an odd prime so $p - 1$ is even and complete graph of even number is very cost effective. Hence, $N(\mathbb{Z}_n)$ is very cost effective graph. \square

THEOREM 3.4. *If p and q are distinct prime number and n is a positive integer, then $\Omega(\mathbb{Z}_n)$ is not very cost effective graph, where $n = p^2q$.*

PROOF. Let $n = p^2q$, where p and q be a distinct primes. Then the non-nilradical elements are all zero-divisors which are not divisible by pq . Since these elements are not adjacent to themselves but the vertices which are multiple of p^2 is adjacent to multiple of q . Here, p is also another vertex which is not adjacent to any other vertices. Therefore, p is an isolated vertices. Hence $\Omega(\mathbb{Z}_n)$ is not very cost effective graph. \square

THEOREM 3.5. *Let p_1, p_2, \dots, p_m be distinct prime and $n = p_1 \cdot p_2 \dots p_m, m \geq 1$. Then $\Omega(\mathbb{Z}_n)$ is very cost effective graph.*

PROOF. Here, $\Omega(\mathbb{Z}_n) = \Gamma(\mathbb{Z}_n)$. So $\Omega(\mathbb{Z}_n)$ is very cost effective graph. \square

4. Bipartition in $T(\Gamma(\mathbb{Z}_n))$

In this section, we have studied the very cost effective properties of $T(\Gamma(\mathbb{Z}_n))$ for $n = 2p$ and $n = pq$.

THEOREM 4.1. *Let $n = 2p$, p be an odd prime. Then the total graph $T(\Gamma(\mathbb{Z}_n))$ is not very cost effective.*

PROOF. If $n = 2p$, then $\Gamma(\mathbb{Z}_n)$ is a star graph with p vertices. Now, total graph of $\Gamma(\mathbb{Z}_n)$ is a graph with $2p - 1$ vertices in which $p - 1$ vertices have degree two, $p - 1$ vertices have degree p and one vertex which is itself p has degree $2p - 2$ respectively. In order to prove $T(\Gamma(\mathbb{Z}_n))$ is not very cost effective, let $T(\Gamma(\mathbb{Z}_n))$ be a very cost effective graph. Then it has a very cost effective bipartition $\pi = \{R, B\}$. Take $u \in R$ and suppose u is a vertex whose degree is 2. Then $|N(u) \cap R| = 0$ and $|N(u) \cap B| = 2$. Now, the vertex $p \in B$ because u is adjacent to the vertex p . So, all the vertices whose degree are 2 belong to the set R because these vertices are adjacent to vertex p . Also, rest of the $p - 1$ vertices which are of the form $[2, p], [2 \cdot 2, p], \dots, [2(p - 1), p]$ will be in B . If any one of these vertices belongs to R , then R is not very

cost effective set. So, take $[u_i, p] \in B$, where $u_i = 2.i$, $1 \leq i \leq p-1$. Then $|N([u_i, p]) \cap B| = p-1$ and $|N([u_i, p]) \cap R| = 1$. Hence, the set B is not very cost effective set, which contradicts our assumption. Thus, the graph $T(\Gamma(\mathbb{Z}_n))$ is not very cost effective. \square

THEOREM 4.2. *Let $n = pq$, $p, q \geq 3$ and $p < q$ are primes. Then total graph $T(\Gamma(\mathbb{Z}_n))$ is very cost effective.*

PROOF. If $n = pq$, then $\Gamma(\mathbb{Z}_n)$ is a bipartite graph with $p + q - 2$ vertices and $(p-1)(q-1)$ edges. Now, in total graph $T(\Gamma(\mathbb{Z}_n))$, there is $pq - 1$ vertices in which $q-1$ vertices are multiples of p and each has degree $2(p-1)$. Also $p-1$ vertices which are multiples of q have degree $2(q-1)$. The vertices $[u_i, v_j] \in V(T(\Gamma(\mathbb{Z}_n)))$, $u_i = p.i$, $1 \leq i \leq q-1$ and $v_j = q.j$, $1 \leq j \leq p-1$ have degree $p+q-2$. Suppose set R contains the element $\{p, 2p, \dots, (q-1)p, [u_1, v_k], [u_2, v_L], [u_3, v_k], \dots, [u_{q-1}, v_L]\}$ and B contains $\{q, 2q, \dots, (p-1)q, [u_1, v_L], [u_2, v_k], [u_3, v_L], \dots, [u_{q-1}, v_k]\}$ where $1 \leq k \leq \frac{p-1}{2}$, $\frac{p+1}{2} \leq L \leq p-1$. Take bipartition $\pi = \{R, B\}$. We have to show that this bipartition is very cost effective.

Now, consider $u_i \in R$, then $|N(u_i) \cap R| = p-2$ and $|N(u_i) \cap B| = p$. Also, for $[u_i, v_j] \in R$, we have $|N([u_i, v_j]) \cap R| = \frac{p+q-4}{2}$ and $|N([u_i, v_j]) \cap B| = \frac{p+q}{2}$. So, for every vertex $v \in R$, $|N(v) \cap R| < |N(v) \cap B|$ and the set R is very cost effective set. In the set B , take $v_j \in B$, then $|N(v_j) \cap B| = q-2$ and $|N(v_j) \cap R| = q$. Again, take $[u_i, v_j] \in B$, we have $|N([u_i, v_j]) \cap B| = \frac{p+q-4}{2}$ and $|N([u_i, v_j]) \cap R| = \frac{p+q}{2}$. Therefore, $|N(v) \cap B| < |N(v) \cap R|$ for every vertex v in B . So set B is also very cost effective. Hence, the bipartition π is very cost effective bipartition and the total graph $T(\Gamma(\mathbb{Z}_n))$ is very cost effective graph. \square

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